

# Bijjective mapping preserving intersecting antichains for $k$ -valued cubes

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## Abstract

Generalizing a result of Miyakawa, Nozaki, Pogosyan and Rosenberg, we prove that there is a one-to-one correspondence between the set of intersecting antichains in a subset of the lower half of the  $k$ -valued  $n$ -cube and the set of intersecting antichains in the  $k$ -valued  $(n - 1)$ -cube.

## 1 Introduction

Let  $k$  and  $n$  be positive integers with  $k \geq 2$ , and let  $E = \{0, \dots, k - 1\}$ . A  $k$ -valued  $n$ -cube is the cartesian power  $E^n$ . Let  $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in E^n$ . We write  $\mathbf{a} \preceq \mathbf{b}$  if  $a_i \leq b_i$  for all  $i \in [n] := \{1, \dots, n\}$ . We call  $A \subseteq E^n$  an *antichain* if there are no different elements  $\mathbf{a}, \mathbf{b}$  of  $A$  such that  $\mathbf{a} \preceq \mathbf{b}$ . A family  $A \subseteq E^n$  is *intersecting* if for all  $\mathbf{a}, \mathbf{b} \in A$  there exists  $i \in [n]$  such that  $a_i + b_i \geq k$ . This is a natural generalization of the binary case ( $k = 2$ ), where the elements of  $E^n$  can be interpreted as the subsets of  $[n]$  and an intersecting antichain is one consisting of pairwise intersecting sets. The restriction in the definition applies also when  $b = a$ , so no  $\mathbf{a} \in E^n$  with  $a_i < \frac{k}{2}$  for all  $i \in [n]$  is an element of any intersecting antichain, because then  $a_i + a_i < k$  for all  $i \in [n]$ .

In the binary case, there is a bijective map from the lower half of the  $n$ -cube onto the  $(n - 1)$ -cube that preserves intersecting antichains in both directions [1]. Answering a question of Miyakawa [2], we present a generalization to the  $k$ -valued case. The proof is slightly simpler than that of [1] for the  $k = 2$  case. More information on intersecting antichains can be found in [3].

The *weight* of an element  $\mathbf{a} \in E^n$ , written  $w(\mathbf{a})$ , is defined by  $w(\mathbf{a}) = a_1 + \dots + a_n$ . For  $0 \leq t \leq n(k-1)$  the  $t$ -th *layer*  $\mathcal{B}_t$  of  $E^n$  is denoted by  $\mathcal{B}_t = \{\mathbf{a} \in E^n : w(\mathbf{a}) = t\}$ . Now we define the "lower half"  $L_n$  with restricted first entries.

Let  $g = \lfloor \frac{n(k-1)}{2} \rfloor$  and notice that  $g = \frac{1}{2}(nk - n - 1)$  if  $n(k-1)$  is odd and  $g = \frac{1}{2}n(k-1)$  otherwise. Let  $C_i = \{(a_1, \dots, a_n) \in E^n : a_1 = i\}$ . Let

$$L_n = \begin{cases} (\mathcal{B}_0 \cup \dots \cup \mathcal{B}_g) \cap (C_0 \cup C_{k-1}) & \text{if } n(k-1) \text{ is odd,} \\ ((\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{g-1}) \cap (C_0 \cup C_{k-1})) \cup (\mathcal{B}_g \cap C_0) & \text{otherwise.} \end{cases}$$

This set can be given also as follows: Let  $g' = \lfloor \frac{n(k-1)-1}{2} \rfloor$ , and notice that  $g' = \frac{1}{2}(nk - n - 1) = g$  if  $n(k-1)$  is odd and  $g' = \frac{1}{2}n(k-1) - 1 = g - 1$  otherwise. Thus

$$L_n = \begin{cases} (\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{g'}) \cap (C_0 \cup C_{k-1}) & \text{if } n(k-1) \text{ is odd,} \\ ((\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{g'}) \cap (C_0 \cup C_{k-1})) \cup (\mathcal{B}_{g'+1} \cap C_0) & \text{otherwise.} \end{cases}$$

Hence,  $g$  is the maximum weight of the elements of  $L_n$  beginning with 0. Similarly,  $g'$  is the maximum weight of the elements of  $L_n$  beginning with  $k-1$ . Notice that  $g + 1 + g' = n(k-1)$ .

## 2 A map from $L_n$ to $E^{n-1}$

For  $a \in E$ , let  $\bar{a} = k-1-a$ . Define a map  $\varphi$  from  $L_n$  into  $E^{n-1}$  by setting

$$\varphi((a_1, \dots, a_n)) = \begin{cases} (a_2, \dots, a_n) & \text{if } a_1 = 0, \\ (\bar{a}_2, \dots, \bar{a}_n) & \text{if } a_1 = k-1. \end{cases}$$

Obviously  $\bar{\bar{a}} = a$  and  $a = b$  iff  $\bar{a} = \bar{b}$ . Concerning the weight  $w$ , note that

$$w(\varphi(\mathbf{a})) = \begin{cases} w(\mathbf{a}) & \text{if } a_1 = 0, \\ (k-1)(n-1) - (w(\mathbf{a}) - (k-1)) & \text{if } a_1 = k-1. \end{cases}$$

**Lemma 1.** For  $\mathbf{a}, \mathbf{b} \in L_n$  with  $a_1 = 0$  and  $b_1 = k-1$ , we have

$$w(\varphi(\mathbf{a})) < w(\varphi(\mathbf{b})).$$

*Proof.* We have

$$\begin{aligned} w(\varphi(\mathbf{b})) &= (k-1)(n-1) - (w(\mathbf{b}) - (k-1)) = n(k-1) - w(\mathbf{b}) \\ &= g + 1 + g' - w(\mathbf{b}) \geq g + 1 \geq w(\mathbf{a}) + 1 = w(\varphi(\mathbf{a})) + 1 \\ &> w(\varphi(\mathbf{a})). \end{aligned}$$

□

**Lemma 2.** *The map  $\varphi$  is injective.*

*Proof.* Let  $\mathbf{a}, \mathbf{b} \in L_n, \mathbf{a} \neq \mathbf{b}$ . If  $a_1 = b_1$ , we obtain immediately from the definition of  $\varphi$  that  $\varphi(\mathbf{a}) \neq \varphi(\mathbf{b})$ . If  $a_1 \neq b_1$ , w.l.o.g.  $a_1 = 0$  and  $b_1 = k - 1$ . By Lemma 1,  $w(\varphi(\mathbf{b})) > w(\varphi(\mathbf{a}))$ , hence  $\varphi(\mathbf{a}) \neq \varphi(\mathbf{b})$ .  $\square$

**Lemma 3.** *The map  $\varphi$  is surjective.*

*Proof.* We have to show that for all  $\mathbf{b} = (b_1, \dots, b_{n-1}) \in E^{n-1}$  there exists an  $\mathbf{a} \in L_n$  such that  $\varphi(\mathbf{a}) = \mathbf{b}$ . We construct this  $\mathbf{a}$  as follows: Let

$$\mathbf{a} = \begin{cases} (0, b_1, \dots, b_{n-1}) & \text{if } w(\mathbf{b}) \leq g, \\ (k-1, \bar{b}_1, \dots, \bar{b}_{n-1}) & \text{if } w(\mathbf{b}) > g. \end{cases}$$

If  $w(\mathbf{b}) \leq g$ , then  $w(\mathbf{a}) = w(\mathbf{b}) \leq g$ . If  $w(\mathbf{b}) > g$ , then  $w(\mathbf{a}) = k-1 + ((k-1)(n-1) - w(\mathbf{b})) < n(k-1) - g = g' + 1$ , hence  $w(\mathbf{a}) \leq g'$ . Thus in both cases  $\mathbf{a} \in L_n$ , and obviously  $\varphi(\mathbf{a}) = \mathbf{b}$ .  $\square$

**Corollary 1.** *The map  $\varphi : L_n \rightarrow E^{n-1}$  is a bijection.*

**Lemma 4.** *Both  $\varphi$  and its inverse preserve intersecting antichains.*

*Proof.* Due to the definition of an intersecting antichain, it is sufficient to prove the lemma for antichains  $A$  with  $|A| \in \{1, 2\}$ .

Let  $\mathbf{a}, \mathbf{b} \in L_n$  and let  $\{\mathbf{a}, \mathbf{b}\}$  be an intersecting antichain.

If  $a_1 = b_1 = 0$ , then obviously  $\{\varphi(\mathbf{a}), \varphi(\mathbf{b})\}$  is an intersecting antichain.

If  $a_1 = b_1 = k-1$ , then

$$\begin{aligned} w(\varphi(\mathbf{a})) + w(\varphi(\mathbf{b})) &= (k-1)(n-1) - (w(\mathbf{a}) - (k-1)) \\ &\quad + (k-1)(n-1) - (w(\mathbf{b}) - (k-1)) \\ &\geq 2n(k-1) - 2 \left\lfloor \frac{n(k-1) - 1}{2} \right\rfloor \\ &> (k-1)(n-1). \end{aligned}$$

Thus, there exists  $i \in \{2, \dots, n\}$  such that  $\bar{a}_i + \bar{b}_i \geq k$ , and hence  $\{\varphi(\mathbf{a}), \varphi(\mathbf{b})\}$  is intersecting. Furthermore, if  $\mathbf{a} = \mathbf{b}$ , obviously  $\{\varphi(\mathbf{a}), \varphi(\mathbf{b})\} = \{\varphi(\mathbf{a})\}$  is an antichain. If  $\mathbf{a} \neq \mathbf{b}$ , by the antichain property, there are  $i, j \in \{2, \dots, n\}$  with  $a_i < b_i$  and  $a_j > b_j$ . Thus  $\bar{a}_i > \bar{b}_i$  and  $\bar{a}_j < \bar{b}_j$ , and hence  $\{\varphi(\mathbf{a}), \varphi(\mathbf{b})\}$  is an antichain.

If  $a_1 \neq b_1$ , then we may assume  $a_1 = 0$  and  $b_1 = k-1$ . Obviously  $\mathbf{a} \neq \mathbf{b}$ . By Lemma 1,  $w(\varphi(\mathbf{a})) < w(\varphi(\mathbf{b}))$ , and thus  $\varphi(\mathbf{a}) \not\preceq \varphi(\mathbf{b})$ . Since  $\{\mathbf{a}, \mathbf{b}\}$  is intersecting, there exists  $i \in \{2, \dots, n\}$ , such that  $a_i + b_i \geq k$ . Thus  $\bar{b}_i = k-1 - b_i < a_i$ , hence  $\varphi(\mathbf{a}) \not\preceq \varphi(\mathbf{b})$ . Consequently  $\{\varphi(\mathbf{a}), \varphi(\mathbf{b})\}$  is

an antichain. Since  $\{\mathbf{a}, \mathbf{b}\}$  is an antichain, there exists  $i \in \{2, \dots, n\}$ , such that  $a_i > b_i$ , so  $a_i + \bar{b}_i = a_i + k - 1 - b_i > k - 1$ , and hence  $\{\varphi(\mathbf{a}), \varphi(\mathbf{b})\}$  is intersecting.

Now let  $\mathbf{a}, \mathbf{b} \in E^{n-1}$  and let  $\{\mathbf{a}, \mathbf{b}\}$  be an intersecting antichain. By the proof of Lemma 3, for  $\mathbf{b} \in E^{n-1}$ ,

$$\varphi^{-1}(\mathbf{b}) = \begin{cases} (0, b_1, \dots, b_{n-1}) & \text{if } w(\mathbf{b}) \leq g, \\ (k-1, \bar{b}_1, \dots, \bar{b}_{n-1}) & \text{if } w(\mathbf{b}) > g. \end{cases}$$

If  $w(\mathbf{a}) \leq g$  and  $w(\mathbf{b}) \leq g$ , then obviously  $\{\varphi^{-1}(\mathbf{a}), \varphi^{-1}(\mathbf{b})\}$  is an intersecting antichain.

If  $w(\mathbf{a}) > g$  and  $w(\mathbf{b}) > g$ , then the first entry of both  $\varphi^{-1}(\mathbf{a})$  and  $\varphi^{-1}(\mathbf{b})$  is  $k-1$ , so  $\{\varphi^{-1}(\mathbf{a}), \varphi^{-1}(\mathbf{b})\}$  is intersecting. Furthermore, if  $\mathbf{a} = \mathbf{b}$ , obviously  $\{\varphi^{-1}(\mathbf{a}), \varphi^{-1}(\mathbf{b})\} = \{\varphi^{-1}(\mathbf{a})\}$  is an antichain. If  $\mathbf{a} \neq \mathbf{b}$ , there are  $i, j \in [n-1]$  with  $a_i < b_i$  and  $a_j > b_j$ , thus  $\bar{a}_i > \bar{b}_i, \bar{a}_j < \bar{b}_j$ , and hence  $\{\varphi^{-1}(\mathbf{a}), \varphi^{-1}(\mathbf{b})\}$  is an antichain.

In the remaining case, we may assume  $w(\mathbf{a}) \leq g$  and  $w(\mathbf{b}) > g$ . Obviously  $\mathbf{a} \neq \mathbf{b}$ . The first entry of  $\varphi^{-1}(\mathbf{a})$  is 0 and the first entry of  $\varphi^{-1}(\mathbf{b})$  is  $k-1$ , so  $\varphi^{-1}(\mathbf{a}) \not\leq \varphi^{-1}(\mathbf{b})$ . Since  $\{\mathbf{a}, \mathbf{b}\}$  is intersecting, there exists  $i \in [n-1]$  such that  $a_i + b_i \geq k$ . Thus  $a_i \geq k - b_i = \bar{b}_i + 1 > \bar{b}_i$ , and hence  $\varphi^{-1}(\mathbf{a}) \not\leq \varphi^{-1}(\mathbf{b})$ . Consequently  $\{\varphi^{-1}(\mathbf{a}), \varphi^{-1}(\mathbf{b})\}$  is an antichain. Since  $\{\mathbf{a}, \mathbf{b}\}$  is an antichain, there exists  $i \in [n-1]$  such that  $a_i > b_i$ , thus  $a_i + \bar{b}_i = a_i + k - 1 - b_i > k - 1$ , hence  $\{\varphi^{-1}(\mathbf{a}), \varphi^{-1}(\mathbf{b})\}$  is intersecting.  $\square$

From Corollary 1 and Lemma 4 we immediately obtain the main result of this note.

**Theorem 1.** *The map  $\varphi$  is bijective and preserves intersecting antichains in both directions.*

### 3 Maximum Size of an Antichain and an Intersecting Antichain in $E^n$ and $L_n$

To show an application of Theorem 1, we first estimate the maximum size of an intersecting antichain in  $E^n$ .

**Theorem 2.** *The map  $\varphi$  is bijective and preserves intersecting antichains in both directions.*

*Proof.* Let  $W$  be a maximum intersecting antichain of size  $m$ . Set

$$s := \min\{t : W \cap \mathcal{B}_t \neq \emptyset\},$$

$$W' := (W^{\succeq} \cap \mathcal{B}_{s+1}) \cup (W \setminus \mathcal{B}_s).$$

A direkt check shows that  $W'$  is an intersecting antichain.

## 4 Remarks

In the definition of  $L_n$ , we can replace  $C_0$  by  $C_i$  and  $C_{k-1}$  by  $C_{k-1-i}$  with  $0 \leq i < \frac{k-1}{2}$ . We obtain

$$L_{n,i} = \begin{cases} (\mathcal{B}_0 \cup \dots \cup \mathcal{B}_g) \cap (C_i \cup C_{k-1-i}) & \text{if } n(k-1) \text{ is odd,} \\ ((\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{g-1}) \cap (C_i \cup C_{k-1-i})) \cup (\mathcal{B}_g \cap C_i) & \text{otherwise.} \end{cases}$$

The analogue on  $L_{n,i}$  of the map  $\varphi$  on  $L_n$  also is a bijection to  $E^{n-1}$  and preserves intersecting antichains. The only place where the proof is not completely identical is the case  $a_1 = b_1 = k-1-i$  in the first direction of Lemma 4. In this case, we have

$$\begin{aligned} w(\varphi(\mathbf{a})) + w(\varphi(\mathbf{b})) &= (k-1)(n-1) - (w(\mathbf{a}) - (k-1-i)) \\ &\quad + (k-1)(n-1) - (w(\mathbf{b}) - (k-1-i)) \\ &\geq 2(n-1)(k-1) - 2 \left\lfloor \frac{n(k-1)-1}{2} \right\rfloor + 2(k-1-i) \\ &> 2(k-1)(n-1) - n(k-1) + (k-1) \\ &= (k-1)(n-1). \end{aligned}$$

Furthermore,  $g$  can be replaced by  $g+z$  and  $g'$  by  $g'-z$  with  $z \in \{0, \dots, g'\}$ , such that

$$L_n^z := ((\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{g+z}) \cap C_0) \cup ((\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{g'-z}) \cap C_{k-1}).$$

As in the definition in Lemma 3, for  $\mathbf{b} \in E^{n-1}$  we have

$$\varphi^{-1}(\mathbf{b}) = \begin{cases} (0, b_1, \dots, b_{n-1}) & \text{if } w(\mathbf{b}) \leq g+z, \\ (k-1, \bar{b}_1, \dots, \bar{b}_{n-1}) & \text{if } w(\mathbf{b}) > g+z. \end{cases}$$

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## References

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